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Holder Estimates for Local Solutions of Some Doubly Nonlinear Degenerate Parabolic Equations

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We prove interior and boundary Holder continuity for bounded weak solutions of a class of quasilinear parabolic equations whose prototype is

$$u_t = \operatorname{div}(|u|^{m-1} |Du|^{p-2} Du).$$

0. INTRODUCTION

One of the results of this note is a fine description of the local behaviour of locally bounded weak solutions of degenerate parabolic equations of the type

$$\begin{aligned} &u \in C_{\mathrm{loc}}(0, T; L^2_{\mathrm{loc}}(\Omega)), \qquad |u|^{(m-1)/p} |Du| \in L^p_{\mathrm{loc}}(\Omega_T) \\ &u_t - \operatorname{div}(|u|^{m-1} |Du|^{p-2} Du) = 0, \qquad m \geq 1, \quad p \geq 2. \end{aligned} \tag{0.1}$$

Here Ω is a domain in R^N with smooth boundary $\partial\Omega$, and for $0 < T < \infty$ we have set $\Omega_T \equiv \Omega \times (0, T)$. We will prove Holder estimates for the solution of (0.1) and general quasilinear version of it. We also will establish regularity results up to the parabolic boundary of Ω_T if the solution takes regular Dirichlet or variational data. We refer to Section 1 for the precise statement of these results.

Here we point out that the interest in this kind of evolution problems

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stems both from their mathematical structure and from a spectrum of application to concrete problems.

Equation (0.1) is degenerate since its modulus of ellipticity vanishes when either the solution u or its spatial gradient Du vanishes. In this sense they are intrinsically degenerate. Their theoretical interests is in the understanding to what extent the principal part contributes to the diffusion.

Special cases are the porous medium equation ($p = 2$ and $m > 1$) and the p-Laplacian equation ($m = 1$ and $p > 2$). These have been widely studied in the literature (see [Ka, Va, DiB2, Ar] and the references therein). On the contrary, results concerning doubly degenerate equations of the type (0.1) are more fragmented. This is due mainly to the inherent difficulty of a double degeneracy. Most of the available results have to do with existence and uniqueness of solution (see, for example, [Ra, TS, Li, Br1, Br2, Ka]).

We refer to the nice survey paper of Kalashnikov [Ka4] for a summary of the state of the art about the qualitative and quantitative properties on this kind of evolution process. Some qualitative information has been recently established in [Va, EV] in one space dimension. To our knowledge however no information is available on the local regularity of the solutions, not even in dimension 1.

As an attempt to unify the theory of the local behaviour of porous media type equations and p-Laplacian type degeneracies, we will establish precise local and global Holder estimates for weak solutions of (0.1). Such a compactness would yield most of the known existence theorems via Ascoli–Arzela theorem and identification of weak limits via Minty's lemma [Mi].

We briefly comment on the application to physics. When $p = 2$ and $m > 1$ the connection with the flow in porous media is by now classical (see [Ba1, Ba2]). When $m \geqslant 1$ and $p > 2$ Equation (0.1) models the non-stationary, polytropic flow of a fluid in a porous medium whose tangential stress has a power dependence of the velocity (non-Newtonian elastic filtration). We refer to [BER] for further information on these phenomena.

Recently a connection has been revealed with soil science; specifically with flows in reservoirs exhibiting fractured media (see [SW]). Further applications can be found in [BM].

The paper is organised as follows. In Section 1 we give the precise statement of the main results of this paper. Section 2 is devoted to some preliminary estimates. In Section 3 we introduce the intrinsic geometry reflecting the degeneracy exibited by the equation. In Section 4 a proposition is stated that implies the regularity results of Section 1. This proposition is proved in Section 9 and is based on an alternative argument. In Sections 5 and 6 the first alternative is analyzed, while the second one is considered in Sections 7 and 8.

1. Main Results

Consider quasi-linear parabolic equations with principal part in divergence form of the type

$$u_t - \operatorname{div} a(x, t, u, Du) = b(x, t, u, Du) \qquad \text{in } \mathscr{D}'(\Omega_T), \tag{1.1}$$

where Ω is a bounded open set in R^N, $0 < T < +\infty$ and $\Omega_T \equiv \Omega \times (0, T)$. The functions $a: \Omega_T \times R^{N+1} \to R^N$ and $b: \Omega_T \times R^{N+1} \to R$, are only assumed to be measurable and to satisfy the structure conditions

$$a(x, t, u, Du)\, Du \geqslant C_0 \Phi(|u|)\, |Du|^p - \psi_0(x, t), \qquad p \geqslant 2 \tag{A_1}$$

$$|a(x, t, u, Du)| \leqslant C_1 \Phi(|u|)\, |Du|^{p-1} + \Phi^{1/p}(|u|)\, \psi_1(x, t), \tag{A_2}$$

$$|b(x, t, u, Du)| \leqslant C_2 \Phi(|u|)\, |Du|^p + \psi_2(x, t), \tag{A_3}$$

where $\Phi(\cdot): R^+ \to R^+$ is continuous and satisfies

$$\begin{aligned} &\gamma_1 s^{\beta_1} \leqslant \Phi(s) \leqslant \gamma_2 s^{\beta_2}, \qquad &&\forall 0 \leqslant s \leqslant \sigma_0,\\ &\Lambda_1 \leqslant \Phi(s) \leqslant \Lambda_2, &&\forall s \geqslant \sigma_0, \end{aligned} \tag{A_4}$$

Here C_i, $i = 1, 2, 3$, β_j, γ_j, Λ_j, $j = 1, 2$, are given positive constants and $\Lambda_1 \leqslant \Lambda_2$ might depend on a local or global sup-bound of the solution. The functions ψ_i, $i = 0, 1, 2$ are non-negative, defined in Ω_T, and subject to the condition

$$\psi_0, \psi_1^{p/(p-1)}, \psi_2 \in L^{q^\wedge, r^\wedge}(\Omega_T), \tag{A_5}$$

where $q^\wedge, r^\wedge \geqslant 1$ satisfy

$$\frac{1}{r^\wedge} + \frac{N}{pq^\wedge} = 1 - \kappa_1, \tag{A_6}$$

and

$$q^\wedge \in [1, \infty], \qquad r^\wedge \in \left[\frac{1}{1-\kappa_1}, \frac{p}{p(1-\kappa_1)-1}\right],$$

$$\kappa_1 \in (0, (p-1)/p) \qquad \text{if} \quad N = 1; \tag{A_6(i)}$$

$$q^\wedge \in [N/(p - p\kappa_1), \infty], \qquad r^\wedge \in [1/(1-\kappa_1), \infty],$$

$$\kappa_1 \in (0, 1) \qquad \text{if} \quad N > 1,\ 1 < p \leqslant N; \tag{A_6(ii)}$$

$$q^\wedge \in [N/(p - p\kappa_1), \infty], \qquad r^\wedge \in [1/(1-\kappa_1), \infty],$$

$$\kappa_1 \in ((p-N)/p, 1), \qquad \text{if} \quad 1 < N < p. \tag{A_6(iii)}$$

A measurable function u is a local weak solution of (1.1) in Ω_T if

$$u \in C_{\mathrm{loc}}(0, T; L^2_{\mathrm{loc}}(\Omega)), \qquad \Phi^{1/(p-1)}(|u|)\,|Du| \in L^p_{\mathrm{loc}}(\Omega_T) \tag{1.2}$$

and for every compact subset K of Ω and for every subinterval $[t_1, t_2]$ of $(0, T]$

$$\int_K u\psi\, dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_K \{-u\psi_t + a(x, \tau, u, Du)\, D\psi\}\, dx\, d\tau$$

$$= \int_{t_1}^{t_2} \int_K b(x, \tau, u, Du)\, \psi\, dx\, d\tau,$$

for all testing funcrtions $\psi \in W^{1,2}_{\mathrm{loc}}(0, T; L^2(K)) \cap L^p_{\mathrm{loc}}(0, T; W^{1,p}_0(K))$.

We assume throughout that

$$u \in L^\infty_{\mathrm{loc}}(\Omega_T) \tag{A$_7$}$$

In theorems below, the statement that a constant γ depends upon the data, means that it can be determined a priori only in dependence of the various constants appearing in (A_1)–(A_4), and the norms $\|\psi_0, \psi_1^{p/(p-1)}, \psi_2\|_{q^\wedge, r^\wedge; \Omega_T}$ where $q^\wedge$, $r^\wedge$ are linked by (A_6). With $\partial\Omega$ we denote the boundary of Ω and set

$$S_T \equiv \partial\Omega \times (0, T]; \qquad \Gamma \equiv S_T \cup \Omega \times \{0\}. \tag{A$_8$}$$

Clearly Γ is the parabolic boundary of Ω_T.

I. *Interior Regularity*

THEOREM 1.1. *Let u be a locally bounded local weak solution of* (1.1), *and let* (A_1)–(A_6) *hold. Then* $(x, t) \to u(x, t)$ *is locally Holder continuous in* Ω_T, *and for every compact subset K of Ω_T, there exists constants $\gamma > 1$ and* $\alpha \in (0, 1)$ *such that*

$$|u(x_1, t_1) - u(x_2, t_2)| \leqslant \gamma(|x_1 - x_2|^\alpha + |t_1 - t_2|^{\alpha/p}),$$

for every pair of points $(x_1, t_1), (x_2, t_2) \in K$.

The constant γ depends only upon the norm $\|u\|_{\infty,K}$, the data and the dist$(K; \Gamma)$. The constant α depends only upon the data and the norm $\|u\|_{\infty,K}$ and is independent of dist$(K; \Gamma)$. Besides, both γ and α are non-decreasing in $\|u\|_{\infty,K}$.

Remark 1.1. In the next sections, we will prove this result by applying a technique due to Di Benedetto [DiB1] via an alternative argument. The main difference between this paper and [DiB1], is that, here, to handle a double degenerate equation it is necessary to introduce a different metric. For this reason, we will focus our attention only on what is new, and refer the reader to [DiB1], when only a simple straightforward modification is necessary.

II. *Boundary Regularity*

(i) *Dirichlet Data.* The boundary $\partial\Omega$ is assumed to satisfy the property of *positive geometric density* (see [LSU]).
Consider the Dirichlet problem

$$\begin{aligned} &u_t - \operatorname{div} a(x, t, u, Du) = b(x, t, u, Du), && \text{in } \mathscr{D}'(\Omega_T) \\ &u(\cdot, t)|_{\partial\Omega} = g(\cdot, t), && \text{a.e. } t \in (0, T) \qquad (1.4) \\ &u(\cdot, 0) = u_0(\cdot). && \end{aligned}$$

On the Dirichlet data g and u_0 we assume

$$g \text{ is continuous on } \bar{S}_T \text{ with modulus of continuity } \omega_g(\cdot) \qquad (A_9)$$

$$u_0 \text{ is continuous in } \bar{\Omega} \text{ with modulus of continuity } \omega_0(\cdot). \qquad (A_{10})$$

THEOREM 1.2. *Let u be a bounded weak solution of the Dirichlet problem* (1.4) *and assume that* (A_1)–(A_{10}) *hold.*

Then $(x, t) \to u(x, t)$ is continuous in the closure $\bar{\Omega}_T$, and there exists a positive non-decreasing function $s \to \omega(s)\colon R^+ \to R^+$, $\omega(0) = 0$, such that

$$|u(x_1, t_1) - u(x_2, t_2)| \leqslant \omega(|x_1 - x_2| + |t_1 - t_2|^{1/p}),$$

for every pair of points $(x_1, t_1), (x_2, t_2) \in \bar{\Omega}_T$. In particular if the boundary datum $(x, t) \to g(x, t)$ is Holder continuous in S_T with exponent say α_1, and if the initial datum is Holder continuous in $\bar{\Omega}$ with exponent say α_2, then $(x, t) \to u(x, t)$ is Holder continuous in $\bar{\Omega}_T$ and there exist constants $\gamma > 1$ and $\alpha \in (0, 1)$ such that

$$|u(x_1, t_1) - u(x_2, t_2)| \leqslant \gamma(|x_1 - x_2|^\alpha - |t_1 - t_2|^{\alpha/p}),$$

for every pair of points $(x_1, t_1), (x_2, t_2) \in \bar{\Omega}_T$.

The constant γ depends only upon the norm $\|u\|_{\infty, \Omega_T}$, and the data. The constant α depends only upon the norm $\|u\|_{\infty, \Omega_T}$, the Holder exponents α_i, $i = 1, 2$ of g and u_0, respectively, and the data. Besides, both γ and α are non-decreasing in $\|u\|_{\infty, \Omega_T}$.

Remark 1.2. The Holder exponent α of u will satisfy

$$\alpha \leqslant \min\{\alpha_1, \alpha_2\}.$$

Moreover the proof will show that α is independent of $\|u\|_{\infty,\Omega_T}$ if in the structure condition (A_3) the constant C_2 is sufficiently small or if (A_3) is replaced by

$$|b(x, t, u, Du)| \leqslant C_2 \Phi(|u|)\,|Du|^{p-1} + \psi_2. \qquad (A_3')$$

We have stated the theorem in a *global way* for simplicity of presentation. Indeed the proof has a *purely local* thrust. For example the boundary datum $(x, t) \to g(x, t)$ could be continuous only on an open portion of S_T.

Analogous remarks hold in the case where u_0 is only locally continuous.

(ii) *Variational Data.* To stress such a *locality* we state our next theorem as if no information were available on the initial datum u_0.

We assume here that $\partial\Omega$ is piecewise smooth, so that the outward unit normal, which we denote with ν, is defined a.e. on $\partial\Omega$.

Consider the Neumann problem

$$\begin{aligned} &u_t - \operatorname{div} a(x, t, u, Du) = b(x, t, u, Du), && \text{in } \Omega_T \\ &a(x, t, u, Du)\cdot \nu = \phi(x, t, u), && \text{on } S_T \\ &u(\cdot, 0) = u_0(x), && \end{aligned} \qquad (1.5)$$

where we assume that the function $\phi(\cdot, t, u(\cdot, t))$ admits, for a.e. $t \in (0, T)$, an extension into Ω which we denote with $\phi^\wedge(\cdot, t, u(\cdot, t))$ such that

$$\begin{aligned} &(|\phi^\wedge| + |\phi^\wedge_u|)^{p/(p-1)} \leqslant \Phi(|u|)^{1/(p-1)}\psi_*, \\ &|D\phi^\wedge| \leqslant \psi_*, \qquad \psi_* \in L^{q^\wedge, r^\wedge}(\Omega_T). \end{aligned} \qquad (A_{11})$$

THEOREM 1.3. *Fix* $\varepsilon \in (0, T]$ *and consider the cylindrical domain* $\Omega \times [\varepsilon, T]$. *Let u be a weak solution of the Neumann problem* (1.5) *satisfying*

$$u \in L^\infty(\bar\Omega \times [\varepsilon, t]),$$

and let (A_1)–(A_5) *and* (A_{11}) *hold.*

Then $(x, t) \to u(x, t)$ *is Holder continuous in* $\bar\Omega \times [\varepsilon, T]$, *for all* $\varepsilon > 0$, *and there exist constants* γ *and* α *such that*

$$|u(x_1, t_1) - u(x_2, t_2)| \leqslant \gamma(|x_1 - x_2|^\alpha + |t_1 - t_2|^{\alpha/p}),$$

for every pair of points $(x_1, t_1), (x_2, t_2) \in \bar\Omega \times [\varepsilon, T]$.

The constant $\gamma > 1$ *and* $\alpha \in (0, 1)$ *depend upon* ε, *the norm* $\|u\|_{\Omega \times [\varepsilon, T]}$ *(in a non-decreasing way), and the data, including the norms appearing in the assumption* (A_{11}).

The constant α *depends upon* ε *only through the norm* $\|u\|_{\infty,\Omega} \times [\varepsilon, T]$.

Remark 1.3. If in the structure condition (A_3) the constant C_2 is sufficiently small or if (A_3) is replaced by (A_3'), then α is independent of $\|u\|_{\infty, \Omega \times [\varepsilon, T]}$, and hence it is independent of ε.

Remark 1.4. The continuity of u can be claimed up to $t = 0$ provided assumption (A_{10}) holds. Also if u_0 is Holder continuous in $\bar{\Omega}$ then u is Holder continuous in $\bar{\Omega}_T$.

Remark 1.5. The proofs of Theorems 1.2 and 1.3 result from obvious modifications of the arguments presented in the proof of interior regularity and so we do not give them.

2. Preliminary Results

We state some integral inequalities in the interior of Ω_T, that will be the main tools in establishing local Holder estimates for the solutions.

Let $\rho > 0$ and $K_\rho \equiv \{x \in R^N \mid \max_{1 \leqslant i \leqslant N} |x_i| < \rho\}$. If $x_0 \in R^N$, let $x_0 + K_\rho$ be the cube of centre x_0 and wedge 2ρ which is congruent to K_ρ. Let θ be a given positive number and consider the cylinder

$$Q(\theta, \rho) \equiv K_\rho \times \{-\theta, 0\}$$

and if $(x_0, t_0) \in R^{N+1}$ we let $(x_0, t_0) + Q(\theta, \rho)$ denote the cylinder with "vertex" at (x_0, t_0) congruent to $Q(\theta, \rho)$.

Fix $(x_0, t_0) \in \Omega_T$ and let ρ and θ be so small that $(x_0, t_0) + Q(\theta, \rho) \subset \Omega_T$. In $(x_0, t_0) + Q(\theta, \rho)$ introduce piecewise smooth cutoff functions $(x, t) \to \zeta(x, t)$ and $x \to \xi(x)$ such that they both satisfy

$$0 \leqslant \zeta \leqslant 1, \qquad |D\zeta| < \infty, \qquad \zeta(x, t) = 0, \qquad \text{for } x \text{ outside } x_0 + K_\rho, \forall t \leqslant t_0. \tag{2.1}$$

Assume that

$$u \in L^\infty_{\text{loc}}(\Omega_T) \cap L^p_{\text{loc}}(0, T; W^{1,p}_{\text{loc}}(\Omega)) \tag{2.2}$$

and construct the truncated functions $(u-k)_\pm$, where we choose levels k satisfying

$$\operatorname*{ess\,sup}_{(x_0, t_0) + Q(\theta, \rho)} |(u-k)_\pm| \equiv H_k^\pm \leqslant \delta, \tag{2.3}$$

where δ is a given positive number.

Remark 2.1. Suppose (2.3) is written for $(u-k)_+$ and assume the number δ is small. Then the levels k are forced to be near the essential sup of u in $(x_0, t_0) + Q(\theta, \rho)$.

Define

$$A_{k,\rho}^\pm(\tau) \equiv \{x \in (x_0 + K_\rho) \mid (u(x, \tau) - k)_\pm > 0\}. \tag{2.4}$$

To estimate the contribution of the *lower order* terms ψ_i, $i=0,1,2$, consider the numbers q, r, κ constructed starting from $q^\wedge$, $r^\wedge$, κ_1, appearing in (A_6), as follows:

$$q=\frac{q^\wedge p(1+\kappa)}{q^\wedge-1}, \qquad r=\frac{r^\wedge p(1+\kappa)}{r^\wedge-1}, \qquad \kappa=\frac{p}{N}\kappa_1. \tag{2.5}$$

It is seen from (A_6), that they satisfy

$$\frac{1}{r}+\frac{N}{pq}=\frac{N}{p^2}, \tag{2.6}$$

and their admissible range is

$$\begin{aligned} &q\in(p,\infty], && r\in[p^2,\infty), && \text{if} \quad N=1 \\ &q\in\left[p,\frac{Np}{N-p}\right], && r\in[p,\infty), && \text{if} \quad 1<p<N \\ &q\in[p,\infty), && r\in\left(\frac{p^2}{N},\infty\right), && \text{if} \quad 1<N\leqslant p. \end{aligned} \tag{2.7}$$

Define also the logarithmic function

$$\Psi(H_k^\pm,(u-k)_\pm,v)\equiv \ln^+\left\{\frac{H_k^\pm}{H_k^\pm-(u-k)_\pm+v}\right\};\ v<\min\{H_k^\pm,1\}. \tag{2.8}$$

PROPOSITION 2.1. *Let u be a locally bounded weak solution of* (1.1) *in Ω_T. Let* (A_1)–(A_6) *hold. There exist constants γ and δ_0 that can be determined a priori only in terms of the data, such that for every cylinder $(x_0,t_0)+Q(\theta,\rho)\subset\Omega_T$ and for every level k satisfying* (2.3) *for $\delta\leqslant\delta_0$*

$$\begin{aligned} &\sup_{t_0-\theta<t<t_0}\int_{x_0+K_\rho}(u-k)_\pm^2\,\zeta^p\,dx \\ &\qquad+\gamma^{-1}\iint_{(x_0,t_0)+Q(\theta,\rho)}\Phi(|u|)\,|D[(u-k)_\pm\,\zeta]|^p\,dx\,d\tau \\ &\leqslant\int_{x_0+K_\rho}(u-k)_\pm^2\,\zeta^p(x,t_0-\theta)\,dx \\ &\qquad+\gamma\iint_{(x_0,t_0)+Q(\theta,\rho)}\Phi(|u|)(u-k)_\pm^p\,|D\zeta|^p\,dx\,d\tau \\ &\qquad+\gamma\iint_{(x_0,t_0)+Q(\theta,\rho)}(u-k)_\pm^2\,\zeta^{p-1}\,\zeta_t\,dx\,d\tau \\ &\qquad+\gamma(\delta_0)\left\{\int_{t_0-\theta}^{t_0}|A_{k,\rho}^\pm(\tau)|^{r/q}\,d\tau\right\}^{p(1+\kappa)/r}, \end{aligned} \tag{2.9}$$

and

$$\sup_{t_0-\theta<t<t_0} \int_{x_0+K_\rho} \Psi^2(H_k^\pm, (u-k)_\pm, v)(x,t)\, \xi^p(x)\, dx$$

$$\leqslant \int_{x_0+K_\rho} \Psi^2(H_k^\pm, (u-k)_\pm, v)(x, t_0-\theta)\, \xi^p(x)\, dx$$

$$+\gamma \iint_{(x_0,t_0)+Q(\theta,\rho)} \Phi(|u|)\, \Psi\, |\Psi_u(H_k^\pm, (u-k)_\pm, v)|^{2-p}\, |D\xi|^p\, dx\, d\tau$$

$$+\frac{\gamma}{v^2}\left(1+\ln\frac{H_k^\pm}{v}\right)\left\{\int_{t_0-\theta}^{t_0} |A_{k,\rho}^\pm(\tau)|^{r/q}\, d\tau\right\}^{p(1+\kappa)/r}. \tag{2.10}$$

Remark 2.2. The proof shows that the number δ_0 in Proposition 2.1 has to be chosen small according to the constant C_2; for example, a good choice could be $\delta_0 = C_0/4C_2$. So if in (A_3), $C_2 = 0$ then δ_0 can be taken to be infinite and no restriction is imposed on the levels k.

3. Interior Regularity: The Intrinsic Geometry

As in [Dib1], the Holder continuity of a solution of (1.1), will be heuristically a consequence of the following fact: for every $(x_0, t_0) \in \Omega_T$ there exists a family of nested and shrinking cylinders $Q(\theta_n, \rho_n)$ with same *vertex* such that the essential oscillation of u in $(x_0, t_0) + Q(\theta_n, \rho_n)$, tends to zero, as $n \to \infty$ in a way that can be "quantitatively" determined by the operator (1.1) and the data in the structure conditions (A_1)–(A_6).

As already said in the introduction, the key idea of the proof is to work with cylinders whose dimensions are suitably rescaled to reflect the degeneracy exhibited by the equation.

To make this precise, fix a point $(x_0, t_0) \in \Omega_T$ and construct the cylinder

$$(x_0, t_0) + Q(R^{p-\varepsilon}, 2R),$$

where ε is a small positive number to be determined later and R is so small that such a cylinder is all contained in Ω_T. After a translation we may assume that $(x_0, t_0) \equiv 0$ and set

$$\mu^+ = \operatorname*{ess\,sup}_{Q(R^{p-\varepsilon}, 2R)} u, \qquad \mu^- = \operatorname*{ess\,inf}_{Q(R^{p-\varepsilon}, 2R)} u. \tag{3.1}$$

Let ω and M be given by

$$\omega = \operatorname*{ess\,osc}_{Q(R^{p-\varepsilon}, 2R)} u \equiv \mu^+ - \mu^-, \qquad M \equiv \max\{\|u\|_{\infty, Q(R^{p-\varepsilon}, 2R)}; \omega\}. \tag{3.2}$$

Thus

$$M = \max\{|\mu^+|, |\mu^-|, \omega\}.$$

Construct the cylinder

$$Q(a_0 R^p, R), \qquad \text{where} \quad \frac{1}{a_0} = \left(\frac{\omega}{A}\right)^{p-2} \Phi(M), \tag{3.3}$$

whre A is a constant to be determined.

We will assume that

$$\left(\frac{\omega}{A}\right)^{p-2} \Phi(M) > R^{\varepsilon}, \tag{3.4}$$

so that we have the inclusion

$$Q(a_0 R^p, R) \subset Q(R^{p-\varepsilon}, 2R), \tag{3.5}$$

and the following inequality is obviously verified

$$\operatorname*{ess\,osc}_{Q(a_0 R^p, R)} u \leqslant \omega. \tag{3.6}$$

By *cylinders rescaled to reflect the degeneracy*, we mean boxes of the type (3.3) where the length has been suitably stretched to accomodate the degeneracy. If $p = 2$ and $\Phi(s) = 1$, then the boxes in (3.3) are the standard parabolic boxes reflecting the natural homogenity of the space and time variables, while if $p > 2$, they are the ones used in [Dib1].

To simplify the presentation we will impose on the function Φ a condition stronger that (A_4), that is

$$\gamma_1 s^{\beta} \leqslant \Phi(s) \leqslant \gamma_2 s^{\beta}, \qquad \forall s \geqslant 0. \tag{3.7}$$

The proof will show that this restriction is not essential.

4. The Main Proposition

PROPOSITION 4.1. *There exist constants ε, $\mu \in (0, 1)$, and $C, A > 1$, that can be determined a priori depending only upon the data and $\|u\|_{\infty, Q(R^{p-\varepsilon}, 2R)}$, satisfying the following. Construct the sequences $\omega_0 = \omega$, $R_0 = R$, $M_0 = M$, and*

$$\begin{aligned} R_n &= \frac{1}{C^n} R, \qquad \omega_{n+1} = \max\{\mu\omega_n; C R_n^{\varepsilon}\} \\ M_n &\equiv \max\{\omega_n; \|u\|_{\infty, Q^{(n-1)}}\}; \qquad n = 1, 2, ..., \end{aligned} \tag{4.1}$$

where

$$Q^{(n)} \equiv Q(a_n R_n^p, R_n) \qquad \textit{and} \qquad \frac{1}{a_n} = \left(\frac{\omega_n}{A}\right)^{p-2} \Phi(M_n). \tag{4.2}$$

Then for all $n = 0, 1, 2, \ldots,$

$$Q^{(n+1)} \subset Q^{(n)} \qquad \textit{and} \qquad \operatorname*{ess\,osc}_{Q^{(n)}} u \leqslant \omega_n. \tag{4.3}$$

A consequence of this Proposition is (see [LSU, Lemma 5.8]) that there exist constants $\gamma > 1$ and $\alpha \in (0, 1)$, that can be determined a priori only in terms of the data and of the norm $\|u\|_{\infty, Q(R^{p-\varepsilon}, 2R)}$, such that for all the cylinders

$$\begin{gathered} 0 < \rho \leqslant R, \qquad Q(a_0 \rho^p, \rho), \qquad \frac{1}{a_0} = \left(\frac{\omega}{A}\right)^{p-2} \Phi(M), \\ \operatorname*{ess\,osc}_{Q(a_0 \rho^p, \rho)} u \leqslant \gamma(\omega + R^{\varepsilon}) \left(\frac{\rho}{R}\right)^{\alpha}. \end{gathered} \tag{4.4}$$

Statements of Holder continuity made in Theorem 1.1 now follow immediately via a standard covering argument.

Remark 4.1. The Proof of Proposition 4.1 will show that indeed it is sufficient to work with numbers ω and cylinders $Q(a_0 R^p, R)$ linked by

$$\operatorname*{ess\,osc}_{Q(a_0 R^p, R)} u \leqslant \omega. \tag{4.5}$$

This fact is in general not verifiable, for a given box, since the dimension of the box would have to be *intrinsically* defined in terms of the essential oscillation of u within it.

Therefore the role of having introduced the cylinder $Q(R^{p-\varepsilon}, 2R)$ and having assumed (3.4) is that (4.5) holds true for the *constructed box* $Q(a_0 R^p, R)$. It will be part of the proof of Proposition 4.1 to show that at each step the cylinders $Q^{(n)}$ and the essential oscillation of u within them, satisfy the intrinsic geometry dictated by (4.5).

To begin the proof, inside $Q(a_0 R^p, R)$ consider subcylinders of *smaller size* constructed as follows. The number ω being fixed, let s_0 be the smallest positive integer such that

$$2 \|u\|_{\infty, Q(R^{p-\varepsilon}, 2R)} / 2^{s_0} \leqslant \delta_0, \tag{4.6}$$

where the number δ_0 is introduced in Proposition 2.1 (local energy estimates). Then construct cylinders

$$(0, \bar{t}) + Q(dR^p, R), \qquad \frac{1}{d} = \left(\frac{\omega}{2^{s_0}}\right)^{p-2} \Phi(M). \tag{4.7}$$

These are contained inside $Q(a_0 R^p, R)$ if the number A is chosen larger than 2^{s_0} and if $\bar{t}$ ranges over

$$0 > \bar{t} > -\{A^{p-2} - (2^{s_0})^{p-2}\} \frac{R^p}{\omega^{p-2}\Phi(M)}.$$

The structure of the proof is based on studying separately two cases. Either we can find a cylinder of the type $(0, t) + Q(dR^p, R)$ where u is *mostly* large, or such a cylinder cannot be found. In either case the conclusion is that the essential oscillation of u in a smaller cylinder about (x_0, t_0) decreases in a way that can be quantitatively measured. In the arguments to follow we assume (3.4) is in force and determine later the numbers A and ε. Also without loss of generality we may assume that

$$\mu^+ = \|u\|_{\infty, Q(R^{p-\varepsilon}, 2R)}, \qquad \text{i.e.,} \qquad \mu^+ \geqslant |\mu^-|, \tag{4.8}$$

so that $M = \max\{\mu^+; \omega\}$.

Remark 4.2. For later use let us compute the quantity

$$G(\omega, R) \equiv \gamma R^{N\kappa} \left(\frac{\omega}{2^{s_0}}\right)^{-2} d^{[p(1+\kappa)]/r}, \tag{4.9}$$

where γ is a constant depending only upon the data and κ is defined in (2.5).

From the definition (3.2) of M, the definition of d in (4.7), and the assumption (3.7) on the function $\Phi(\cdot)$, it follows that

$$G(\omega, R) \leqslant A_1 R^{N\kappa} \omega^{-b}, \qquad \text{where} \quad b = 2 + [(p-2) + \beta]^{p(1+\kappa)/r} \tag{4.10}$$

and

$$A_1 = \gamma A^{2 + (p-2)(p(1+\kappa)/r)} \gamma_1^{(p(1+\kappa)/r)}.$$

In the proof we will encounter quantities of the type $A_i R^{N\kappa} \omega^{-b}$, $i = 1, 2, \ldots, l$, where A_i are constants that can be determined a priori only in terms of the data and are independent of ω and R. We may assume without loss of generality that they satisfy

$$A_i R^{N\kappa} \omega^{-b} \leqslant 1. \tag{4.11}$$

Indeed if not, we would have $\omega \leqslant C R^{\varepsilon}$ for the choices

$$C = \max_{1 \leqslant i \leqslant l} A_i^{1/b} \qquad \text{and} \qquad \varepsilon = \frac{N\kappa}{b},$$

and the proposition would be trivial.

In the estimates to follow we denote with γ a generic positive constant that can be calculated a priori depending only upon the data and that may be different in different contexts.

5. The First Alternative

Lemma 5.1. *There exists a number* $\nu_0 \in (0, 1)$ *independent of* ω, R, A *such that if for some cylinder of the type* $(0, \bar{t}) + Q(dR^p, R)$

$$\left| (x, t) \in [(0, \bar{t}) + Q(dR^p, R)] \mid u(x, t) < \mu^- + \frac{\omega}{2^{s_0}} \right| \leqslant \nu_0 \, |Q(dR^p, R)|$$

then

$$u(x, t) > \mu^- + \frac{\omega}{2^{s_0 + 4}} \qquad \textit{for a.e.} \quad (x, t) \in \left[(0, \bar{t}) + Q\left(d\left(\frac{R}{2}\right)^p, \frac{R}{2} \right) \right]. \tag{5.1}$$

Proof. Fix a cylinder for which the assumption of the lemma holds and change variables so that $(0, \bar{t}) \equiv (0, 0)$, and we can work within cylinders $Q(d\rho^p, \rho)$, $0 < \rho \leqslant R$. Let

$$R_n = \frac{R}{2} + \frac{R}{2^{n+1}}, \qquad n = 0, 1, 2, ...,$$

construct the family of nested cylinders $Q(dR_n^p, R_n)$ and let ζ_n be a piecewise smooth cutoff function in $Q(dR_n^p, R_n)$ such that

$$\begin{aligned}
&0 \leqslant \zeta_n(x, t) \leqslant 1, \qquad \forall (x, t) \in Q(dR_n^p, R_n), \\
&\zeta_n \equiv 1 \qquad \text{in } Q(dR_{n+1}^p, R_{n+1}); \\
&\zeta_n = 0 \qquad \text{on the parabolic boundary of } Q(dR_n^p, R_n) \\
&|D\zeta_n| \leqslant \frac{2^{n+1}}{R}, \qquad 0 \leqslant \frac{\partial}{\partial t} \zeta_n \leqslant \frac{2^{p(n+1)}}{dR^p} \\
&\frac{1}{d} = \left(\frac{\omega}{2^{s_0}}\right)^{p-2} \Phi(M).
\end{aligned} \tag{5.2}$$

We will use the inequality (2.9) written over the cylinders $Q(dR_n^p, R)$, for the functions $(u - k_n)_-$, where for $n = 0, 1, 2, ...$

$$\begin{aligned}
k_n &= \mu^- + \frac{\omega}{2^{s_0+1}} + \frac{\omega}{2^{s_0+1+n}}, \qquad \text{if} \quad \mu^- + \frac{\omega}{2^{s_0+1}} \geqslant \frac{\omega}{2^{s_0+2}} \\
k_n &= \mu^- + \frac{\omega}{2^{s_0+4}} + \frac{\omega}{2^{s_0+n+4}}, \qquad \text{if} \quad \mu^- + \frac{\omega}{2^{s_0+1}} < \frac{\omega}{2^{s_0+1}} < \frac{\omega}{2^{s_0+2}}.
\end{aligned} \tag{5.3}$$

This choice of levels is justified since

$$\operatorname*{ess\,sup}_{Q(dR_n^p,\, R_n)} (u-k_n)_- \leqslant \frac{\omega}{2^{s_0}} \leqslant \delta_0.$$

In this setting (2.9) takes the form

$$\begin{aligned}
&\operatorname*{ess\,sup}_{-dR_n^p<t<0} \int_{K_{R_n}} (u-k_n)_-^2\, \zeta_n^p(x,t)\,dx \\
&\quad + \iint_{Q(dR_n^p,\, R_n)} \Phi(|u|)\,|D[(u-k_n)_-\,\zeta_n]|^p\,dx\,d\tau \\
&\leqslant \gamma\frac{2^{np}}{R^p}\left\{\iint_{Q(dR_n^p,\, R_n)} \Phi(|u|)(u-k_n)_-^p\,dx\,d\tau\right. \\
&\quad \left. + \frac{1}{d}\iint_{Q(dR_n^p,\, R_n)} (u-k_n)_-^2\,dx\,d\tau\right\} \\
&\quad + \gamma\left\{\int_{-dR_n^p}^{0} |A_{k_n, R_n}^-|^{r/q}\,d\tau\right\}^{p(1+\kappa)/r}.
\end{aligned} \tag{5.4}$$

We will show that as $n\to\infty$

$$\iint_{Q(dR_n^p,\, R_n)} \chi[(u-k_n)_->0]\,dx\,d\tau \to 0.$$

Since $k_n \to k_\infty \geqslant \mu^- + \omega/2^{s_0+4}$ this would imply that

$$\left|\left[u<\mu^- + \frac{\omega}{2^{s_0+4}}\right] \cap Q\left(d\left(\frac{R}{2}\right)^p, \frac{R}{2}\right)\right| = 0$$

thereby proving the lemma.

The arguments below have the purpose of finding comparable upper and lower bounds for $\Phi(|u|)$ within the set $[u<k_n]$.

If on such a set u is away from zero then $\Phi(|u|)$ is away from zero and the equation is only degenerate because $p>2$. We treat first the least favourable case in which u might be near zero.

Assume first that

$$\mu^- + \frac{\omega}{2^{s_0+1}} \geqslant \frac{\omega}{2^{s_0+2}} \tag{5.5}$$

and choose the levels k_n according to (5.3). Introduce the truncated functions

$$w = \begin{cases} u & \text{if } \ u>\omega/2^{s_0+4} \\ \omega/2^{s_0+4} & \text{if } \ u\leqslant \omega/2^{s_0+4}, \end{cases}$$

and observe that $(u-k_n)_+ \geqslant (w-k_n)_+$ and

$$\int_{K_{R_n}} (u-k_n)_+^2 \; \zeta_n^p(x,t)\,dx \geqslant \left(\frac{2^{s_0}}{\omega}\right)^{p-2} \int_{K_{R_n}} (w-k_n)_+^p \; \zeta_n^p(x,t)\,dx.$$

Moreover

$$\begin{aligned} &\iint_{Q(dR_n^p,\,R_n)} \Phi(|u|)\,|D[(u-k_n)_+ \; \zeta_n]|^p \,dx\,d\tau \\ &\quad\geqslant \iint_{Q(dR_n^p,\,R_n)} \Phi(w)\,|D[(w-k_n)_+ \; \zeta_n]|^p \,dx\,d\tau \\ &\qquad - \frac{\gamma 2^{np}}{R^p} \iint_{Q(dR_n^p,\,R_n)} \Phi(|u|)(u-k_n)_+^p \,dx\,d\tau. \end{aligned} \tag{5.6}$$

If $\mu^- \leqslant \frac{1}{2}\mu^+$, we have $\frac{1}{2}\omega \leqslant \mu^+ \leqslant 2\omega$ so that M is comparable to both ω and μ^+. In such a case on the set $[(w-k_n)_+ > 0]$,

$$\gamma^{-1}(s_0)\,\Phi(M) \leqslant \Phi(w) \leqslant \gamma(s_0)\,\Phi(M). \tag{5.7}$$

If $\mu^- > \frac{1}{2}\mu^+$, then $2\omega < \mu^+ = M$ and on the set $[(w-k_n)_+ > 0]$ we have

$$\Phi(w) \leqslant \Phi(\mu^- + \omega/2^{s_0}) \leqslant \Phi(M)$$

$$\Phi(w) \geqslant \Phi(\mu^-) \geqslant \Phi(\tfrac{1}{2}\mu^+) \geqslant \gamma^{-1}\Phi(M)$$

for a constant γ depending only upon the data. Therefore (5.7) holds in either cases and we can estimate below the first integral on the right hand side of (5.6) by

$$\gamma^{-1}(s_0)\,\Phi(M) \iint_{Q(dR_n^p,\,R_n)} |D(w-k_n)_+ \; \zeta_n|^p \,dx\,d\tau.$$

We conclude that in the case (5.5) holds, the left hand side of (5.4) is estimated below by

$$\begin{aligned} &\left(\frac{\omega}{2^{s_0}}\right)^{2-p} \operatorname*{ess\,sup}_{-dR_n^p < t < 0} \int_{K_{R_n}} (w-k_n)_+^p \; \zeta_n^p(x,t)\,dx \\ &\quad + \gamma^{-1}(s_0) \left(\frac{\omega}{2^{s_0}}\right)^{2-p} \left(\frac{\omega}{2^{s_0}}\right)^{p-2} \Phi(M) \\ &\quad \times \iint_{Q(dR_n^p,\,R_n)} |D[(w-k_n)_+ \; \zeta_n]|^p \,dx\,d\tau \\ &\quad - \frac{\gamma^{2np}}{R^p} \left(\frac{\omega}{2^{s_0}}\right)^2 \left[\left(\frac{\omega}{2^{s_0}}\right)^{p-2} \Phi(M)\right] \\ &\quad \times \iint_{Q(dR_n^p,\,R_n)} \chi[(u-k_n)_+ > 0]\,dx\,d\tau \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{\omega}{2^{s_0}}\right)^{2-p} \operatorname*{ess\,sup}_{dR_n^p < t < 0} \int_{K_{R_n}} (w-k_n)_-^p \zeta_n^p(x,t)\,dx \\
&\quad + \gamma^{-1}(s_0)\left(\frac{\omega}{2^{s_0}}\right)^{2-p} \frac{1}{d} \iint_{Q(dR_n^p, R_n)} |D[(w-k_n)_- \zeta_n]|^p \,dx\,d\tau \\
&\quad - \frac{\gamma 2^{np}}{R^p}\left(\frac{\omega}{2^{s_0}}\right)^2 \left[\left(\frac{\omega}{2^{s_0}}\right)^{p-2} \Phi(M)\right] \\
&\quad \times \iint_{Q(dR_n^p, R_n)} \chi[(u-k_n)_- > 0]\,dx\,d\tau, \qquad (5.8)
\end{aligned}
$$

where we have used the definition (4.7) of d. We proceed to estimate the right hand side of (5.4). We have

$$
\begin{aligned}
&\gamma \frac{2^{np}}{R^p}\left\{\iint_{(Q(dR_n^p, R_n)} \Phi(|u|)(u-k_n)_-^p \,dx\,d\tau + \frac{1}{d}\iint_{Q(dR_n^p, R_n)} (u-k_n)_-^2 \,dx\,d\tau\right\} \\
&\quad \leqslant \gamma \frac{2^{np}}{R^p}\left(\frac{\omega}{2^{s_0}}\right)^2 \left(\frac{\omega}{2^{s_0}}\right)^{p-2} \Phi(M) \iint_{Q(dR_n^p, R_n)} \chi[(w-k_n)_- > 0]\,dx\,d\tau \\
&\quad\quad + \gamma \frac{2^{np}}{R^p}\frac{1}{d}\left(\frac{\omega}{2^{s_0}}\right)^2 \iint_{Q(dR_n^p, R_n)} \chi[(w-k_n)_- > 0]\,dx\,d\tau \\
&\quad = \gamma \frac{2^{np}}{dR^p}\left(\frac{\omega}{2^{s_0}}\right)^2 \iint_{Q(dR_n^p, R_n)} \chi[(w-k_n)_- > 0]\,dx\,d\tau, \qquad (5.9)
\end{aligned}
$$

where we have used again the definition (4.7) of d. Combining the estimates (5.8) and (5.9) as parts of the energy inequalities (5.4) we arrive at

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{dR_n^p < t < 0} \int_{K_{R_n}} (w-k_n)_-^p \zeta_n^p(x,t)\,dx \\
&\quad + \frac{\gamma^{-1}(s_0)}{d} \iint_{Q(dR_n^p, R_n)} |D[(w-k_n)_- \zeta_n]|^p \,dx\,d\tau \\
&\leqslant \gamma \frac{2^{np}}{dR^p}\left(\frac{\omega}{2^{s_0}}\right)^p \iint_{Q(dR_n^p, R_n)} \chi[(w-k_n)_- > 0]\,dx\,d\tau \\
&\quad + \gamma \left(\frac{\omega}{2^{s_0}}\right)^{p-2} d^{p(1+\kappa)/r} \left\{\frac{1}{d}\int_{dR_n^p}^{0} |A_{k_n, R_n}|^{r/q}\,d\tau\right\}^{p(1+\kappa)/r}. \qquad (5.10)
\end{aligned}
$$

In (5.10) we introduce the change of time-variables $z = t/d$ which transforms $Q(dR_n^p, R_n)$ into

$$
Q_n \equiv Q(R_n^p, R_n) \equiv K_{R_n} \times \{-R_n^p, 0\}.
$$

Setting also

$$v(\cdot, z) = u(\cdot, zd) \qquad \text{and} \qquad \zeta_n^{\wedge}(\cdot, zd),$$

inequality (5.10) can be written more concisely as

$$\begin{aligned}
&\|(v-k_n)_- \zeta_n^{\wedge}\|_{V^p(Q_n)}^p \\
&\leqslant \gamma(s_0)\frac{2^{np}}{R^p}\left(\frac{\omega}{2^{s_0}}\right)^p |A_n| \\
&+ \gamma(s_0)\left(\frac{\omega}{2^{s_0}}\right)^{p-2} d^{p(1+\kappa)/r}\left\{\int_{-R_n^p}^0 |A_n(z)|^{r/q}\,dz\right\}^{p(1+\kappa)/r},
\end{aligned} \tag{5.11}$$

where we have set

$$A(z) \equiv \{x \in K_{R_n} \mid v(x, z) < k_n\} \qquad \text{and} \qquad |A_n| = \int_{-R_n}^0 |A(z)|\,dz.$$

Since $(v-k_n)_- \zeta_n^{\wedge}$ vanishes on the lateral boundary of Q_n, using an obvious generalization of (3.7), [LSU, p. 76] we have

$$\begin{aligned}
\|(v-k_n)_-\|_{p,Q_{n+1}}^p &\leqslant \|(v-k_n)_- \zeta_n^{\wedge}\|_{p,Q_n}^p \\
&\leqslant |A_n|^{p/(N+p)} \|(v-k_n)_- \zeta_n^{\wedge}\|_{V^p(Q_n)}^p.
\end{aligned} \tag{5.12}$$

The left hand side of (5.12) is estimated below by

$$\|(v-k_n)_-\|_{p,Q_{n+1}}^p \geqslant |k_n - k_{n+1}|^p\,|A_{n+1}| \geqslant \frac{1}{2^{p(n+2)}}\left(\frac{\omega}{2^{s_0}}\right)^p |A_{n+1}|.$$

Combining these estimates gives

$$\begin{aligned}
|A_{n+1}| &\leqslant \gamma(s_0)\,4^{np}\,\frac{|A_n|^{1+p/(N+p)}}{R^p} \\
&+ \gamma(s_0)\,4^{np}\left(\frac{\omega}{2^{s_0}}\right)^{-2} d^{p(1+\kappa)/r} \\
&\times |A_n|^{p/(N+p)}\left\{\int_{R_n^p}^0 |A_n|^{r/q}\,dz\right\}^{p(1+\kappa)/r}.
\end{aligned} \tag{5.13}$$

Divide by $|Q_{n+1}|$ and introduce the quantities

$$Y_n = \frac{|A_n|}{|Q_n|}, \qquad Z_n = \frac{1}{|K_{R_n}|}\left(\int_{-R_n^p}^0 |A_n(z)|^{r/q}\,dz\right)^{p/r}. \tag{5.14}$$

Using also the fact that, by virtue of Remark 4.2, $R^{N\kappa}(\omega/2^{s_0})^{-2} \cdot d^{p(1+\kappa)/r} \leqslant 1$, we obtain from (5.13) in dimensionless form

$$Y_{n+1} \leqslant \gamma(s_0)\, 4^{np}\{ Y_n^{1+p/(N+p)} + Y_n^{p/(N+p)} Z_n^{1+\kappa} \},$$

for all $n = 0, 1, 2, \ldots$. Next by an easy generalization of (3.8), [LSU, p. 77]

$$\begin{aligned} Z_{n+1}(k_n - k_{n+1})^p &\leqslant |K_{R_{n+1}}|^{-1} \|(v-k_n)_-\|^p_{q,r;Q_{n+1}} \\ &\leqslant |K_{R_{n+1}}|^{-1} \|(v-k_n)_- \zeta_n^{\wedge}\|^p_{q,r;Q_{n+1}} \\ &\leqslant \gamma R^{-N} \|(v-k_n)_- \zeta_n^{\wedge}\|^p_{V^p(Q_n)}, \end{aligned}$$

and using (5.11) we obtain $\forall n = 0, 1, 2, \ldots$

$$Z_{n+1} \leqslant \gamma(s_0)\, 4^{np}\{ Y_n + Z_n^{1+\kappa} \}. \tag{5.16}$$

From an easy generalization of [LSU, p. 96, Lemma 5.7 it follows that Y_n and Z_n tend to zero as $n \to \infty$, provided

$$Y_0 + Z_0^{1+\kappa} \leqslant [4\gamma(s_0)]^{-(1+\kappa)/\theta_0}\, 4^{-p(1+\kappa)/\theta_0^2} \equiv v_0,$$

where $\theta_0 = \min\{p/(N+p); \kappa\}$.

Therefore the lemma follows in the case (5.5) holds. Assume now (5.5) is violated, i.e.,

$$\mu^- + \frac{\omega}{2^{s_0+1}} < \frac{\omega}{2^{s_0+2}}. \tag{5.17}$$

This implies that $\mu^- < 0$ and since we are also assuming (4.8) we must have $\mu^+ \leqslant \omega$. Choosing the levels k_n according to (5.3) we have

$$\mu^- + \frac{\omega}{2^{s_0+4}} + \frac{\omega}{2^{s_0+4+n}} < -\frac{\omega}{2^{s_0+3}}.$$

Therefore on the set $[(u-k_n)_- > 0]$

$$\Phi(|u|) \geqslant \Phi\left(\frac{\omega}{2^{s_0+3}}\right) \geqslant \gamma(s_0)\, \Phi(M).$$

Also by virtue (4.8), $\Phi(|u|) \leqslant \Phi(|\mu^-|) \leqslant \Phi(M)$. It follows that $\Phi(|u|)$ can be estimated above and below by $\Phi(M)$ up to a constant depending only upon s_0 and the proof can be repeated step by step as before. We note that this case is in fact simpler since it is not necessary to introduce the truncated function w.

6. The First Alternative Concluded

Suppose the assumptions of Lemma 5.1 are verified for some box $(0, \bar{t}) + Q(dR^p, R)$. We will exploit the fact that at time level

$$-\theta = \bar{t} - \left[\left(\frac{\omega}{2^{s_0}}\right)^{p-2} \Phi(M)\right]^{-1} \left(\frac{R}{2}\right)^p, \tag{6.1}$$

the function $x \to u(x, t)$ is strictly above the level $\mu^- + \omega/2^{s_0+4}$, in the cube $K_{R/2}$.

Set $\rho = R/2$ and construct the cylinder

$$Q(\theta, \rho) \equiv K_\rho \times (-\theta, 0). \tag{6.2}$$

The next lemma asserts that, owing to (5.1), the set where $u(\cdot, t)$ is close to μ^-, within the smaller cube $K_{\rho/2}$, can be made arbitrarily small for all the time levels $-\theta \leqslant t \leqslant 0$.

Lemma 6.1. *For every number $\nu_1 \in (0, 1)$, there exists a positive integer s_1, depending only upon the data and $\|u\|_{\infty, Q(R^p, 2R)}$ and independent of ω, M, R, such that*

$$\left|x \in K_{R/4} \mid u(x, t) < \mu^- + \frac{\omega}{2^{s_1}}\right| \leqslant \nu_1 \, |K_{R/2}|, \tag{6.3}$$

for all $t \in (-\theta, 0)$.

Even if the proof is quite similar to the one of [Dib 1, Lemma 3.2] we prefer to give it, because there are some non-trivial modifications.

Proof. Consider the logarithmic estimates (2.10), written over the cylinder $Q(\theta, \rho)$, for $(u-k)_-$, $k = \mu^- + \omega/2^{s_0+4}$. As a number ν in the definition (2.8) of Ψ, we take

$$\nu = \frac{\omega}{2^{s_0+4+n}}, \qquad n > 1,$$

where n is a positive number to be chosen. Thus we take

$$\Psi \equiv \ln^+ \left\{\frac{H_k^-}{H_k^- - (u - (\mu^- + \omega/2^{s_0+4}))_- + \omega/2^{s_0+4+n}}\right\},$$

where

$$H_k^- = \operatorname*{ess\,sup}_{Q(\theta, \rho)} \left(u - \left(\mu^- + \frac{\omega}{2^{s_0+4}}\right)\right)_- \leqslant \frac{\omega}{2^{s_0+4}}. \tag{6.4}$$

We observe that, for $t=-\theta$, we have

$$\left(u-\left(\mu+\frac{\omega}{2^{s_0+4}}\right)\right)=0, \qquad \text{and therefore} \quad \Psi(x,-\theta)=0, \quad \forall x\in K_\rho.$$

Combining these remarks in (2.10), we obtain

$$\begin{aligned}\int_{K_\rho}&\Psi^2(x,t)\,\xi^p(x)\,dx\\ &\leqslant\frac{\gamma}{\rho^p}\iint_{Q(\theta,\rho)}\Phi(|u|)\,\Psi\,|\Psi_u|^{2-p}\,dx\,d\tau\\ &\quad+\gamma\left(\frac{\omega}{2^{s_0+4+n}}\right)^{-2}\left(1+\ln H_k^-\left(\frac{\omega}{2^{s_0+4+n}}\right)^{-1}\right)\\ &\quad\times\left(\int_{-\theta}^0|A_{k,\rho}^-(\tau)|^{r/q}\,d\tau\right)^{p(1+\kappa)/r},\end{aligned}$$

where $A_{k,\rho}(\cdot)$ is defined in (2.4) and $x\to\xi(x)$ is a piecewise smooth cutoff function in K_ρ that equals one on $K_{\rho/2}$, and such that $|D\xi|\leqslant 4\rho$.

We proceed to estimate the various terms on the right hand side of (6.5). By (4.8) and (3.2) on the set $[(u-k)_->0]$ we have $\Phi(|u|)\leqslant\Phi(\mu^+)\leqslant\Phi(M)$. Next

$$\Psi\leqslant\ln\left(\frac{\omega/2^{s_0+4}}{\omega/2^{s_0+4+n}}\right)=n\ln 2,$$

and

$$|\Psi_u|^{2-p}=\left|H_k^- -(u-k)_- +\frac{\omega}{2^{s_0+4+n}}\right|^{p-2}\leqslant\left(\frac{\omega}{2^{s_0}}\right)^{p-2}.$$

Therefore the first term on the right hand side of (6.5) is estimated above by

$$\frac{\gamma}{\rho^p}n\theta\left(\frac{\omega}{2^{s_0}}\right)^{p-2}\Phi(M)|K_{\rho/2}|.$$

In view of (6.1) we finally obtain

$$\frac{\gamma}{\rho^p}\iint_{Q(\theta,\rho)}\Phi(|u|)\,\Psi\,|\Psi_u|^{2-p}\,dx\,d\tau\leqslant\gamma nA^{p-2}|K_{\rho/2}|,$$

where γ and A are constants depending only upon the data, and A has to be determined later.

The second term is estimated by using the remarks above and employing the conditions (2.6), (2.7) linking the parameters r, q, κ.

This gives

$$\gamma\left(\frac{\omega}{2^{s_0+4+n}}\right)^{-2}\left(1+\ln H_k^-\left(\frac{\omega}{2^{s_0+4+n}}\right)^{-1}\right)$$
$$\times\left(\int_{-\theta}^{0}|A_{k,\rho}^-(\tau)|\,d\tau\right)^{p(1+\kappa)/r}$$
$$\leqslant \gamma n\left(\frac{\omega}{2^{s_0+4+n}}\right)^{-2}\left[\left(\frac{\omega}{A}\right)^{p-2}\Phi(M)\right]^{p(1+\kappa)/r} R^{N\kappa}\,|K_{\rho/2}|.$$

The number n will be determined shortly, depending only upon the data and independent of ω, M, ρ. Therefore by virtue of Remark 4.2, we may estimate

$$\left(\frac{\omega}{2^{s_0+4+n}}\right)^{-2}\left[\left(\frac{\omega}{A}\right)^{p-2}\Phi(M)\right]^{-p(1+\kappa)/r} R^{N\kappa}\leqslant 1.$$

Combining these remarks into (6.5), yields

$$\int_{K_{\rho/2}}\Psi^2(x,t)\,dx\leqslant \gamma nA^{p-2}\,|K_{\rho/2}|, \tag{6.6}$$

where we have used the fact $\xi\equiv 1$ on $K_{\rho/2}$. We estimate below the integral in (6.6), by extending the integration to the smaller set

$$\left\{x\in K_{\rho/2}\mid u(x,t)<\mu^-+\frac{\omega}{2^{s_0+4+n}}\right\},\qquad t\in(-\theta,0). \tag{6.7}$$

On such a set

$$\Psi^2\geqslant \ln^2\left\{\frac{H_k^-}{H_k^- -\omega/2^{s_0+4}+\omega/2^{s_0+4+n}}\right\},$$

the right hand side of this inequality is a decreasing function of H_k^-. Therefore by virtue of (6.4) we have

$$\Psi^2\geqslant \ln^2\left(\frac{\omega/2^{s_0+5}}{\omega/2^{s_0+4+n}}\right)=(n-1)^2\ln^2 2.$$

Putting this into (6.6), gives that for all $t\in(-\theta,0)$

$$\left|x\in K_{\rho/2}\mid u(x,t)<\mu^-+\frac{\omega}{2^{s_0+4+n}}\right|\leqslant \gamma A^{p-2}\frac{n}{(n-1)^2}\,|K_{\rho/2}|.$$

To prove the lemma we have only to choose n sufficiently large.

The information in Lemma 6.1, will imply that u is strictly bounded away from μ^- in a smaller cylinder.

We continue to assume that the assumptions of Lemma 5.1 are verified for some subcylinder of the type $(0, \bar{t}) + Q(dR^p, R)$, so that the conclusion of Lemma 6.1 is in force.

LEMMA 6.2. *The numbers $\nu_1 \in (0, 1)$ and $s_1 \geqslant 1$ can be chosen a priori only dependent upon the data and the norm $\|u\|_{\infty, Q(R^{p-\varepsilon}, 2R)}$ and independent of ω, M, R, so that*

$$u(x, t) > \mu^- + \omega/2^{s_1+5}, \qquad a.e. \quad (x, t) \in Q(\theta, R/8). \tag{6.8}$$

We do not give the proof of this lemma, because it is based on the argument of [Dib1, Lemma 3.3] and the technique of Lemma 5.1.

We summarise the results obtained so far.

PROPOSITION 6.1. *There exist numbers $\nu_0, \eta_0 \in (0, 1)$ and $A \geqslant 1$ depending only upon the data and the norm $\|u\|_{\infty, Q(R^{p-\varepsilon}, 2R)}$ and independent of ω, M, R, such that if for some cylinder of the type $(0, \bar{t}) + Q(dR^p, R)$*

$$\left| (x, t) \in ((0, \bar{t}) + Q(dR^p, R)) \,|\, u(x, t) < \mu^- + \frac{\omega}{2^{s_0}} \right| \leqslant \nu_0 \,|Q(dR^p, R)| \tag{6.9}$$

then either

$$\omega \leqslant A_1 R^{N\kappa/b}, \tag{6.10}$$

or

$$\operatorname*{ess\,osc}_{Q(d(R/2)^p, R/8)} u \leqslant \eta_0 \omega, \tag{6.11}$$

where b is introduced in Remark 4.2.

Proof. Assume (6.10) is violated. If a cylinder $(0, \bar{t}) + Q(dR^p, R)$ satisfying (6.9) can be found, then by Lemma 6.2, we can determine a positive number s_1 such that

$$\operatorname*{ess\,inf}_{Q(\theta, \rho_0/2)} u \geqslant \mu^- + \frac{\omega}{2^{s_1+5}}, \tag{6.12}$$

where θ is defined in (6.1), with $\rho = R/2$ and where $\rho_0 = R/4$. Change the sign of (6.12) and add $\operatorname{ess\,sup}_{Q(\theta, \rho_0/2)} u$ to the left hand side and μ^+ to the right hand side. This gives

$$\operatorname*{ess\,osc}_{Q(\theta, R/8)} u \leqslant (1 - 1/2^{s_1+5})\, \omega.$$

Therefore the proposition follows with $\eta_0 = (1 - 1/2^{s_1+5})$, since

$$Q(d(R/2)^p, R/8) \subset Q(\theta, R/8).$$

7. The Second Alternative

We assume in this section that the assumption of Lemma 5.1 is violated, i.e., for every subcylinder $(0, \bar{t}) + Q(dR^p, R)$

$$|(x, t) \in ((0, \bar{t}) + Q(dR^p, R)) | u(x, t) < \mu^- + \omega/2^{s_0}| > \nu_0 \, |Q(dR^p, R)|.$$

Since

$$\mu^+ - \omega/2^{s_0} \geqslant \mu^- + \omega/2^{s_0}, \qquad \forall s_0 \geqslant 2,$$

we will rewrite this as

$$|(x, t) \in ((0, \bar{t}) + Q(dR^p, R)) | \, u(x, t) > \mu^+ - \omega/2^{s_0} \, | \leqslant (1 - \nu_0) | \, Q(dR^p, R)| \tag{7.1}$$

valid for all cylinders

$$(0, \bar{t}) + Q(dR^p, R) \subset Q(a_0 R^p, R), \qquad \text{where} \quad 1/a_0 = (\omega/A)^{p-2} \, \Phi(M).$$

In view of (7.1) we will study the behaviour of u near its supremum μ^+ and will be working with the *truncated functions* $(u-k)_+$ for the levels

$$k = \mu^+ - \frac{\omega}{2^{s_0 + i}}, \qquad i \geqslant 0.$$

We will assume (4.8) so that $\mu^+ > \omega/2^{s_0+1}$ and the number $M \equiv \max\{\mu^+ ; \omega\}$ satisfies $\mu^+ \leqslant M \leqslant 2\mu^+$. Therefore in the estimates below, within the sets $[u > \mu^+ - \omega/2^{s_0+i}]$, $\forall \, i \geqslant 0$, we will have

$$\Phi(\mu^+) \leqslant \Phi(M) \leqslant \gamma \Phi(\mu^+), \tag{7.2}$$

for a constant γ depending only upon the constant γ_1, γ_2, and β introduced in (A_4) and independent of ω, M, R.

Lemma 7.1. *Let* $(0, \bar{t}) + Q(dR^p, R) \subset Q(a_0 R^p, R)$ *be fixed and let* (7.1) *hold. There exists a time level*

$$t^* \in \left[\bar{t} - dR^p, \, \bar{t} - \frac{\nu_0}{2} dR^p \right],$$

such that

$$\left| x \in K_R \mid u(x, t^*) > \mu^+ - \frac{\omega}{2^{s_0}} \right| \leqslant \left(\frac{1 - \nu_0}{1 - \nu_0/2} \right) |K_R|.$$

The proof of this estimate can be found in [Dib1, Lemma 4.1].

The lemma asserts that at some time level t^* the set where u is close to its supremum occupies only a portion of the cube K_R. The next lemma claims that this indeed occurs for all time levels in a small interval near the top of the cylinder $(0, \bar{t}) + Q(dR^p, R)$.

LEMMA 7.2. *There exists a positive integer $s_2 > s_0$ such that*

$$|x \in K_R \mid u(x, t) > \mu^+ - \omega/2^{s_2}| \leqslant (1 - (v_0/2)^2) \, |K_R|, \tag{7.3}$$

for all $t \in [\bar{t} - (v_0/2)\, dR^p, \bar{t}]$.

Proof. Consider the logarithmic inequality (2.10) written over the box $K_R \times (t^*, \bar{t})$ for the function $(u-k)_+$ for the levels $k = \mu^+ - \omega/2^{s_0}$. As for the number ν in the definition (2.8) of the function Ψ, we take

$$\nu = \frac{\omega}{2^{s_0+n}}, \qquad n > 0 \text{ to be chosen.}$$

we obtain

$$\Psi = \ln^+ \left\{ \frac{H_k}{H_k^+ - (u - (\mu^+ - \omega/2^{s_0}))_+ + \omega/2^{s_0+n}} \right\}, \tag{7.4}$$

where

$$H_k^+ \equiv \operatorname*{ess\,sup}_{(0,t)+Q(dR^p,R)} (u - (\mu^+ - \omega/2^{s_0}))_+ .$$

The cutoff function $x \to \xi(x)$ is taken so that $0 \leqslant \xi \leqslant 1$, $\xi = 1$ in the cube $K_{(1-\sigma)R}$, $\sigma \in (0, 1)$ and $|D\xi| \leqslant (\sigma R)^{-1}$.

With these choices inequality (2.10) yields

$$\begin{aligned}
\int_{K_{(1-\sigma)R}} \Psi^2(x, t)\, dx &\leqslant \int_{K_R} \Psi^2(x, t^*)\, dx \\
&\quad + \frac{\gamma}{(\sigma R)^p} \int_{t^*}^{t} \int_{K_R} \Phi(|u|)\, \Psi \, |\Psi_u|^{2-p} \\
&\quad + \gamma \left(\frac{\omega}{2^{s_0+n}} \right)^{-2} \left[1 + \ln H_k^+ \left(\frac{\omega}{2^{s_0+n}} \right)^{-1} \right] \\
&\quad \times \left\{ \int_{t^*}^{t} |A_k^+(\tau)|^{r/q}\, d\tau \right\}^{p(1+\kappa)/r},
\end{aligned} \tag{7.5}$$

for all $t \in (t^*, \bar{t})$. The various terms in (7.5) are estimated as follows. First

$$\Psi \leqslant n \ln 2; \qquad |\Psi_u|^{2-p} \leqslant 2^{p-2} \left(\frac{\omega}{2^{s_0}} \right)^{p-2}; \qquad \left[\ln H_k^+ \left(\frac{\omega}{2^{s_0+n}} \right)^{-1} \right] \leqslant n \ln 2.$$

Next, from the definition (7.4) it follows that Ψ vanishes on the set $[u < \mu^+ - \omega/2^{s_0}]$. Therefore using Lemma 7.1, the first integral on the right hand side of (7.5) is estimated above by

$$\int_{K_R} \Psi^2(x, t^*)\, dx \leqslant n^2 \ln^2 2 \left(\frac{1-\nu_0}{1-\nu_0/2}\right) |K_R|.$$

Since $\bar{t} - t^* \leqslant dR^p$, and d is given by (4.7), the second integral is estimated by

$$\frac{\gamma}{(\sigma R)^p} \int_{t^*}^{t} \int_{K_R} \Phi(|u|)\, \Psi\, |\Psi_u|^{2-p}\, dx\, d\tau \leqslant \frac{\gamma}{\sigma^p} n\, |K_R|.$$

Finally for the last term, we have the estimate

$$\gamma \left(\frac{\omega}{2^{s_0+n}}\right)^{-2} \left[1 + \ln H_k^+ \left(\frac{\omega}{2^{s_0+n}}\right)^{-1}\right] \left\{\int_{t^*}^{t} |A_k^+(\tau)|^{r/q}\, d\tau\right\}^{p(1+\kappa)/r}$$
$$\leqslant \gamma A_2\, \omega^{-b} R^{n\kappa}\, |K_R|,$$

where $A_2 = 2^{(s_0+n)b}$ and b is defined in (4.10). By Remark 4.2 we may assume that $A_2 \omega^{-b} R^{N\kappa} \leqslant 1$. Combining this remark with (7.5) we conclude that for all $t \in (t^*, \bar{t})$

$$\int_{K_{(1-\sigma)R}} \Psi^2(x, t)\, dx \leqslant n^2 \ln^2 2 \left(\frac{1-\nu_0}{1-\nu_0/2}\right) |K_R| + \frac{\gamma}{\sigma^p} n\, |K_R|. \tag{7.6}$$

The left hand side of (7.6) is estimated below by integrating over the smaller set

$$\left\{x \in K_{(1-\sigma)R} \,|\, u(x, t) > \mu^+ - \frac{\omega}{2^{s_0+n}}\right\}.$$

On such a set, since the function Ψ in (7.4) is decreasing with H_k^+ we have

$$\Psi^2 \geqslant \ln^2 \left(\frac{\omega/2^{s_0+1}}{\omega/2^{s_0+n-1}}\right) = (n-2)^2 \ln^2 2.$$

We carry this in (7.6) and divide through by $(n-2)^2 \ln^2 2$ to obtain

$$\left|x \in K_{(1-\sigma)R} \,|\, u(x, t) > \mu^+ - \frac{\omega}{2^{s_0+n}}\right| \leqslant \left(\frac{n}{n-2}\right)^2 \left(\frac{1-\nu_0}{1-\nu_0/2}\right) |K_R| + \frac{\gamma}{\sigma^p n} |K_R|.$$

On the other hand,

$$\left|x \in K_R \mid u(x, t) > \mu^+ - \frac{\omega}{2^{s_0+n}}\right|$$

$$\leqslant \left|x \in K_{(1-\sigma)R} \mid u(x, t) > v^+ - \frac{\omega}{2^{s_0+n}}\right| + |K_R \setminus K_{(1-\sigma)R}|$$

$$\leqslant \left|x \in K_{(1-\sigma)R} \mid u(x, t) > \mu^+ - \frac{\omega}{2^{s_0+n}}\right| + N\sigma\, |K_R|.$$

Therefore

$$\left|x \in K_R \mid u(x, t) > \mu^+ - \frac{\omega}{2^{s_0+n}}\right| \leqslant \left[\left(\frac{n}{n-2}\right)^2 \left(\frac{1-v_0}{1-v_0/2}\right) + \frac{\gamma}{\sigma^p n} + N\sigma\right] |K_R|,$$

for all $t \in (t^*, \bar{t})$. Choose σ so small that $\sigma N \leqslant \frac{3}{8} v_0^2$ and then n so large that

$$\left(\frac{n}{n-2}\right)^2 \leqslant (1 - v_0/2)(1 + v_0); \qquad \frac{\gamma}{\sigma^p n} \leqslant \frac{3}{8} v_0^2.$$

Then for such a choice of n the lemma follows with $s_2 = s_0 + n$.

Remark 7.1. Since the number v_0 is independent of ω, R, and M also s_2 is independent of these parameters. The number A that determines the length of $Q(a_0 R^p, R)$ is still to be chosen. We will determine it later independent of ω, M, and R, subject to the condition $A > 2^{s_2}$.

Since (7.1) holds for all cylinders of the type $(0, \bar{t}) + Q(dR^p, R)$ the conclusion of Lemma 7.2 holds true for all time levels satisfying

$$t \geqslant (a_0 - d)\, R^p = \left(1 - \left(\frac{2^{s_0}}{A}\right)\right) a_0 R^p,$$

where a_0 and d are defined in (3.3) and (4.7) respectively. If the number A is chosen sufficiently large, we deduce

COROLLARY 7.1. *For all* $t \in (-(a_0/2)\, R^p, 0)$,

$$\left|x \in K_R \mid u(x, t) > \mu^+ - \frac{\omega}{2^{s_2}}\right| \leqslant \left(1 - \left(\frac{v_0}{2}\right)^2\right) |K_R|. \tag{7.7}$$

From now on we will focus on the cylinder $Q((a_0/2)\, R^p, R)$ and to simplify the symbolism we set

$$A_s(t) = \left\{x \in K_R \mid u(x, t) > \mu^+ - \frac{\omega}{2^s}\right\}$$

$$A_s = \left\{(x, t) \in Q\left(\frac{a_0}{2} R^p, R\right) \mid u(x, t) > \mu^+ - \frac{\omega}{2^s}\right\}.$$

The information of Corollary 7.1 will be employed to deduce that the set where u is close to its supremum μ^+, within the cylinder $Q((a_0/2)\, R^p, R)$, can be made arbitrarily small. In this section we will also determine the length of the cylinder $Q(a_0 R^p, R)$ by determining the number A.

LEMMA 7.3. *For every* $\sigma_* \in (0, 1)$ *there exists a number* $s_* > s_2$ *independent of* ω, M, R, *such that*

$$|A_{s_*}| \leqslant \sigma_* \left| Q\left(\frac{a_0}{2} R^p, R\right)\right|. \tag{7.8}$$

Remark 7.2. Assume for the moment that the number s_* has been chosen. Then we determine the length of the cylinder $Q(a_0 R^p, R)$ by choosing

$$A \equiv 2^{s_*}. \tag{7.9}$$

Proof. Consider the local energy estimate (2.9) written over the box $Q(a_0 R^p, 2R)$ for the functions $(u-k)_+$. The levels k are given by

$$k = \mu^+ - \omega/2^s,$$

where $s_2 \leqslant s \leqslant s_*$ andn s_* is to be chosen.

We take a cutoff function ζ that equals one on $Q((a_0/2)\, R^p, R)$, vanishes on the parabolic boundary of $Q((a_0/2)\, R^p, 2R)$ and such that

$$|D\zeta| \leqslant 1/R; \qquad 0 \leqslant \zeta_t \leqslant 1/a_0 R^p.$$

Absorbing the first term on the left hand side of (2.9) in the first term on the right and using the indicated choices we obtain

$$\begin{aligned} &\iint_{Q(a_0/2\, R^p,\, R)} \Phi(|u|)\, |D(u-k)_+|^p \, dx\, d\tau \\ &\quad \leqslant \frac{\gamma}{R^p} \iint_{Q(a_0 R^p,\, 2R)} \Phi(|u|)(u-k)_+^p \, dx\, d\tau \\ &\qquad + \frac{\gamma}{a_0 R^p} \iint_{Q(a_0 R^p,\, 2R)} (u-k)_+^2 \, dx\, d\tau + \gamma \left\{ \int_{a_0 R^p}^0 |A_{k,2R}^+|^{r/q} \right\}^{p(1+\kappa)/r}. \end{aligned} \tag{7.10}$$

We estimate the various term on the right hand side of (7.10) as follows. First using (7.2)

$$\text{(i)} \qquad \begin{aligned} &\frac{\gamma}{R^p} \iint_{Q(a_0 R^p,\, 2R)} \Phi(|u|)(u-k)_+^p \, dx\, d\tau \\ &\quad \leqslant \frac{\gamma}{R^p} \Phi(M) \left(\frac{\omega}{2^s}\right)^p \left| Q\left(\frac{a_0}{2} R^p, R\right)\right|. \end{aligned}$$

Next by virtue of the choice (7.9) of the parameter A, and the definition (3.3) of a_0,

$$\text{(ii)} \qquad \frac{\gamma}{a_0 R^p} \iint_{Q(a_0 R^p, 2R)} (u-k)_+^2 \, dx \, d\tau \leqslant \frac{\gamma}{R^p} \Phi(M) \left(\frac{\omega}{2^s}\right)^p \left| Q\left(\frac{a_0}{2} R^p, R\right) \right|.$$

Finally making use of Remark 4.2 we have

$$\text{(iii)} \quad \gamma \left\{ \int_{-a_0 R^p}^{0} |A_{k,2R}^+|^{r/q} \right\}^{p(1+\kappa)/r} \leqslant \frac{\gamma}{R^p} \Phi(M) \left(\frac{\omega}{2^s}\right)^p \left| Q\left(\frac{a_0}{2} R^p, R\right) \right| \cdot (A_3 \omega^{-b} R^{N\kappa})$$
$$\leqslant \frac{\gamma}{R^p} \Phi(M) \left(\frac{\omega}{2^s}\right)^p \left| Q\left(\frac{a_0}{2} R^p, R\right) \right|,$$

where $A_3 = 2^{bs_*}$ and b is defined in (4.10). We carry these estimates into (7.10) and divide through by $\Phi(M)$ to obtain

$$\iint_{A_s} |Du|^p \, dx \, d\tau \leqslant \frac{\gamma}{R^p} \left(\frac{\omega}{2^s}\right)^p \left| Q\left(\frac{a_0}{2} R^p, R\right) \right|. \tag{7.11}$$

Next we use a lemma of De Giorgi (see [5]) applied to the function $u(\cdot, t)$ for all times $a R^p \leqslant t_0 \leqslant 0$, and for the levels

$$k = \mu^+ - \frac{\omega}{2^s}, \qquad l = \mu^+ - \frac{\omega}{2^{s+1}}, \qquad (l-k) = \frac{\omega}{2^{s+1}}.$$

Note that by virtue of Corollary 7.1 we have

$$\left| x \in K_R \mid u(x, t) < \mu^+ - \frac{\omega}{2^s} \right| \equiv |K_R| - |A_s(t)| \geqslant \left(\frac{v_0}{2}\right)^2 |K_R|.$$

These remarks into the lemma of De Giorgi yield

$$\frac{\omega}{2^{s+1}} |A_{s+1}(t)| \leqslant \frac{4\gamma}{v_0^2} \frac{R^{N+1}}{|K_R|} \int_{A_s(t) - A_{s+1}(t)} |Du| \, dx, \tag{7.12}$$

for all $t \in (-(a_0/2) R^p, 0)$. From this, integrating over such a time interval we obtain

$$\frac{\omega}{2^{s+1}} |A_{s+1}| \leqslant \frac{\gamma}{v_0^2} R \iint_{A_s - A_{s+1}} |Du| \, dx \, d\tau$$
$$\leqslant \frac{\gamma}{v_0^2} R \left(\iint_{A_s} |Du|^p \right)^{1/p} |A_s - A_{s+1}|^{(p-1)/p}.$$

Take the $p/(p-1)$ power, estimate the integral on the right hand side by (7.11), and divide through by $(\omega/2^{s+1})^{p/(p-1)}$. This gives

$$|A_{s+1}|^{p/(p-1)} \leqslant \gamma(v_0)^{-2p/(p-1)} |Q((a_0/2) R^p, R)|^{1/(p-1)} |A_s - A_{s+1}|.$$

These inequalities are valid for all $s_2 \leqslant s \leqslant s_*$. We add them for $s = s_2 + 1$, $s = s_2 + 2, ..., s_*$. The right hand side can be majorized by a convergent series bounded above by $|Q((a_0/2) R^p, R)|$. Therefore we obtain

$$(s_* - s_2 - 1)|A_{s_*}|^{p/(p-1)} \leqslant \gamma(v_0)^{-2p/(p-1)} \left| Q\left(\frac{a_0}{2} R^p, R\right)\right|^{p/(p-1)}.$$

To prove the lemma we divide by $(s_* - s_2 - 1)$ and take s_* so large that

$$\frac{\gamma}{v_0^2 (s_* - s_2 - 1)^{(p-1)/p}} \leqslant \sigma_*.$$

Remark 7.3. If σ_* is independent of ω, M, R, also s_* and hence A are independent of these quantities.

Remark 7.4. The process described in Lemma 7.3 has a double aim. Given σ_* determines a level $\mu^+ - \omega/2^{s_*}$ and a cylinder so that the measure where u is above such a level can be made smaller than σ_*, on that particular cylinder.

8. The Second Alternative Concluded

We may now show that indeed u is strictly below its supremum μ^+ in a smaller box coaxial with $Q((a_0/2) R^p, R)$ and with the same vertex. To simplify the symbolism let us set

$$a_* = \frac{1}{2} a_0 = \left[2\left(\frac{\omega}{A}\right)^{p-2} \Phi(M)\right]^{-1}, \tag{8.1}$$

and write accordingly $Q((a_0/2) R^p, R) \equiv Q(a_* R^p, R)$.

LEMMA 8.1. *The number σ_* (and hence s_* and A) can be chosen so that*

$$u(x, t) \leqslant \mu^+ - \frac{\omega}{2^{s_*+1}}, \qquad a.e.\ Q\left(a_*\left(\frac{R}{2}\right)^p, \frac{R}{2}\right). \tag{8.2}$$

The proof is omitted because it is very similar to the proof of [DiB1, Lemma 4.5] following the same issue of Lemma 6.1.

The following proposition summarises the results of the second alternative and it is proved arguing as in the proof of Proposition 6.1.

PROPOSITION 8.1. *There exist numbers* $\nu_0, \eta_1 \in (0, 1)$, *and* $A_2 \gg 1$ *depending only upon the data and the norm* $\|u\|_{\infty, Q(R^{p-\varepsilon}, 2R)}$, *and independent of* ω, M, R, *such that if for all the cylinders of the type* $(0, \bar{t}) + Q(dR^p, R)$

$$\left|(x, t) \in ((0, \bar{t}) + Q(dR^p, R)) \mid u(x, t) > \mu^+ - \frac{\omega}{2^{s_0}}\right| \leqslant (1 - \nu_0) |Q(dR^p, R)| \tag{8.3}$$

then either

$$\omega \leqslant A_2 R^{N\kappa/b}, \tag{8.4}$$

or

$$\operatorname*{ess\,osc}_{Q(a_*(R/2)^p, R/2)} u \leqslant \eta_1 \omega, \tag{8.5}$$

where b is introduced in Remark 4.2.

9. PROOF OF PROPOSITION 4.1

The two alternatives just discussed, can be combined to prove the main Proposition 4.1. Let us recall that

$$\frac{1}{d} = \left(\frac{\omega}{2^{s_0}}\right)^{p-2} \Phi(M), \qquad \frac{1}{a_0} = \left(\frac{\omega}{A}\right)^{p-2} \Phi(M). \tag{9.1}$$

The concluding statement of the first alternative is that, starting from the cylinder

$$Q(R^{p-\varepsilon}, 2R)$$

and going down to the smaller cylinder

$$Q(d(R/2)^p, R/8)$$

the essential oscillation ω decreases by a factor of $\eta_0 \in (0, 1)$, unless $\omega \leqslant A_1 R^{N\kappa/b}$, where A_1 is a large constant that can be computed a priori only in terms of the data and of the norm $\|u\|_{\infty, Q(R^{p-\varepsilon}, 2R)}$.

Analogously, the conclusion of the second alternative is that starting from the same cylinder and going down to the smaller box $Q(a_0/2(R/2)^p, R/2)$, the number ω decreases of a factor $\eta_1 \in (0, 1)$, unless $\omega \leqslant A_2 R^{N\kappa/b}$, where A_2 is a constant that can be computed a priori in terms of the data.

We combine these two facts into

LEMMA 9.1. *There exist constants*

$$\eta = \max\{\eta_0; \eta_1\} \qquad \text{and} \qquad A = \max\{A_1; A_2\},$$

that can be determined a priori only in terms of the data, such that either

$$\omega \leqslant A R^{N\kappa/b} \qquad \text{or} \qquad \operatorname*{ess\,osc}_{Q(d(R/2)^p,\, R/8)} u \leqslant \eta\omega.$$

To prove the lemma we have only to observe that $Q(d(R/2)^p, R/8)$ is contained in $Q((a_0/2)(R/2)^p, R/2)$.

We comment further on the content of Remark 4.1. The arguments presented, indeed do not require that the starting cylinder be $Q(R^{p-\varepsilon}, 2R)$. It would have been sufficient to have started from the box

$$Q(a_0 R^p, R)$$

if we had known a priori that

$$\operatorname*{ess\,osc}_{Q(a_0 R^p,\, R)} u \leqslant \omega, \tag{9.2}$$

and

$$M = \max\{\omega; \operatorname*{ess\,sup}_{Q(R^{p-\varepsilon},\, 2R)} |u|\}. \tag{9.3}$$

Next we will construct a box for which information of the type (9.2) and (9.3) can be derived. Set

$$\omega_1 \equiv \max\{\eta\omega; A R^{N\kappa/b}\}, \tag{9.4}$$

$$M_1 \equiv \max\{\omega_1; \operatorname*{ess\,sup}_{Q(a_0 R^p,\, R)} |u|\}, \tag{9.5}$$

$$\frac{1}{a_1} = \left(\frac{\omega_1}{A}\right)^{p-2} \Phi(M_1). \tag{9.6}$$

LEMMA 9.2. *There exists a constant γ that can be computed a priori only in terms of the data, such that*

$$\Phi(M) \leqslant \gamma \Phi(M_1). \tag{9.7}$$

Proof. If $M = \omega$ it follows from (9.4) and the structure (A_4) of the function $\Phi(\cdot)$, that

$$\Phi(M) = \Phi(\omega) \leqslant \Phi\left(\frac{1}{\eta}\omega_1\right) \leqslant \frac{\gamma_2}{\gamma_1 \eta^\beta} \Phi(M_1).$$

If $M=\mu^+$ then either $\mu^- \leqslant \frac{1}{2}\mu^+$ or $\mu^- > \frac{1}{2}\mu^+$. In the former case we have $\frac{1}{2}\omega \leqslant \mu^+ \leqslant 2\omega$ and the proof can be concluded as before. In the latter, by the definition of μ^- we have

$$\operatorname*{ess\,sup}_{Q(a_0 R^p,\, R)} u \geqslant \mu^- > \tfrac{1}{2}\mu^+.$$

Therefore $\Phi(M) \leqslant \Phi(2M_1) \leqslant \gamma\Phi(M_1)$.

Let us now return to the cylinder $Q(dR^p/2^p, R/8)$, for which the conclusion of Lemma 9.1 holds, and let us estimate below its length as follows.

$$\begin{aligned} d\left(\frac{R}{2}\right)^p &= \left(\frac{2^{s_0}}{\omega}\right)^{p-2} \frac{1}{\Phi(M)}\left(\frac{R}{2}\right)^p \\ &\geqslant \gamma^{-1}\frac{\eta^{p-2}}{2^p}\left(\frac{2^{s_0}}{A}\right)^{p-2}\left(\frac{A}{\omega_1}\right)^{p-2}\frac{1}{\Phi(M_1)}R^p \geqslant a_1 R_1^p, \end{aligned}$$

where a_1 is as in (9.6) and

$$R_1 = C^{-1}R, \qquad C = \max\left\{\frac{8^{1/p}}{\eta^{(p-2)/p}}\gamma^{1/p}\left(\frac{A}{2^{s_0}}\right)^{(p-2)/p}, 8\right\}.$$

It follows that for the cylinder $Q(a_1 R_1^p, R_1)$ the inequalities (9.2) and (9.3) are verified. Therefore the process can now be repeated starting from such a box, thereby proving Proposition 4.1.

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Note added in proof. Independently, similar results were proved by A. V. Ivanov in "Uniform Holder estimates for weak solutions of quasilinear degenerate parabolic equations" LOMI preprints E-10-1989 (1989) 1-22.

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